**Indian Statistical Institute, Bangalore**

B. Math. Second Year

First Semester - Analysis III

Back Paper Exam  Duration : 3 hours  Max Marks 100  Date : Dec 26, 2016

Remark: Each question carries 20 marks.

1. Let $U, V, W$ be open subsets of $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^l$ respectively, and let $g : W \to V$ and $f : V \to U$ be $C^2$ functions. Let $x_i, y_j$ and $z_k$ denote the co-ordinate functions on $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^l$ respectively. Thus, $1 \leq i \leq n, 1 \leq j \leq m$ and $1 \leq k \leq l$.
   (a) Show that $\frac{\partial}{\partial z_k}(x_i\ o\ f\ o\ g) = \sum_{j=1}^{m} \frac{\partial}{\partial y_j}(x_i o f)\ og\ \frac{\partial}{\partial z_k}(y_j o g)$.
   (b) Define the pull-back $f^*(w)$ of differential forms $w$ on $U$ (under $f$) and show that pull back commutes with exterior derivative, i.e., if $w$ is a differential form on $U$ of class $C^1$ then $f^*(dw) = d(f^*(w))$.
2. Let $U$ and $V$ be open subsets of $R^m$ and $R^n$ and let $f : U \to V$ be a $C^1$ function.
   (a) Show that $f$ is locally Lipscitz, i.e., every point $x\ \varepsilon\ U$ has a neighborhood $K \subseteq U$ and a constant $c > 0$ (depending on $K$) such that $\|f(y) - f(z)\| \leq c\|y - z\|$ for all $y, z\ \varepsilon\ K$.
   (b) If $m = n$ and $x\ \varepsilon\ U$ is such that $f'(x)$ is invertible then, show that there is a neighborhood $K' \subseteq U$ of $x$ and a constant $c' > 0$ such that $\|f(y) - f(z)\| \geq c'\|y - z\|$ for all $y, z\ \varepsilon\ K'$.
3. Show that any two norms on $\mathbb{R}^n$ determine the same notion of convergence of sequences.
4. (a) Give an example of a differentiable function $f$ on a neighborhood $U$ of 0 in $\mathbb{R}$ to itself such that $f'(0) \neq 0, f'$ is bounded on $U$, but $f$ is not one-one on any neighborhood of 0.
   (b) Give an example of a $C^1$ function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f'(x)$ is invertible for all $x$, but $f$ is not one-one.
5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $x\ \varepsilon\ \mathbb{R}^n$.
   (a) If the $n$ partial derivatives of $f$ exist and are bounded on some neighborhood of $x$ then show that $f$ is continuous at $x$.
   (b) If $f$ is differentiable on a neighborhood of $x$, and $f$ attains a local maximum at $x$ then show that $f'(x) = 0$.